

Renormalization group approach to spin glass systems

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Abstract. A renormalization group transformation suitable for spin glass models and, more generally, for disordered models, is presented. The procedure is non-standard in both the nature of the additional interactions and the coarse graining transformation, that is performed on the overlap probability measure. Universality classes are thus naturally defined on a large set of models, going from \mathbb{Z}_2 and Gaussian spin glasses to Ising and fully frustrated models, and others. The proposed analysis is tested numerically on the \mathbb{Z}_2 Edwards–Anderson model in $d = 4$. Good estimates of the critical index ν and of T_c are obtained, and an RG flow diagram is sketched for the first time.

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1 Introduction

In the understanding of critical phenomena a crucial role is played by the renormalization group (RG) analysis [1, 2]. It provides a theoretical foundation of the universality principle and of the scaling theory, as well as a method for a direct evaluation of critical indices. Despite the fact that both universality and scaling are largely used in the study of spin glass (SG) models, the RG approach has not been fully developed (see however [5–7]). Classical tools of RG analysis are not suitable for spin glasses, mainly because a direct transformation of the Hamiltonian is impossible. The random nature of the couplings require also the additional interactions to be introduced as random variables. But averaging over disorder makes it impossible to consider simple Boltzmann–Gibbs distributions, and to use the spin variables for a coarse graining transformation.

Therefore we propose a RG transformation on the overlap distribution, in the spirit of the probabilistic interpretation of the renormalization group [2]. Universality classes are naturally defined on a large set of models, going from \mathbb{Z}_2 and Gaussian spin glasses to Ising and fully frustrated models, and others.

2 The model

A large part of the discussion will be general, but to be explicit let us consider the Edwards–Anderson spin glass

model [3]. Let σ_x be Ising spins located at the sites of a d -dimensional cubic lattice ($x \in \Lambda \subset \mathbb{Z}^d$). The Hamiltonian of the model is given by

$$H(J, \sigma) = - \sum_{\langle x, y \rangle} J_{xy} \sigma_x \sigma_y, \quad (1)$$

where the sum is over the couples of nearest neighbouring sites. Some kind of boundary conditions (*e.g.*, periodic) are also provided. The quenched disordered interactions J_{xy} 's are independent random variables with zero mean and unit variance. Denoting by E expectation on J variables, we have

$$E(J_{xy}) = 0, \quad E(J_{xy}^2) = 1. \quad (2)$$

A Boltzmann–Gibbs measure on the spin variables is introduced, following the usual rules of statistical mechanics. It will be denoted by angular brackets $\langle \cdot \rangle$. Expectation on the disorder is taken only after Boltzmann averages are calculated, and the thermodynamic limit for the appropriate quantities is eventually taken afterwards. The model is symmetric under the *gauge* transformation defined by

$$\begin{cases} J_{xy} \rightarrow J'_{xy} = \varepsilon_x \varepsilon_y J_{xy} \\ \sigma_x \rightarrow \sigma'_x = \varepsilon_x \sigma_x \end{cases} \quad (3)$$

where $\varepsilon_x = \pm 1$ are the gauge group parameters. Let us introduce the overlap variables. Consider s replicas (copies) of the spin variables $\sigma^{(a)}$, $a = 1, \dots, s$. The Hamiltonian of the replicated system is given by $\mathcal{H} = \sum_a H(J, \sigma^{(a)})$, therefore replicas are independent from each other, but

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feel the *same* disorder configuration J . For any couple of replicas (a, b) , introduce the *site overlap* $q_x^{(a,b)}$

$$q_x^{(a,b)} = \sigma_x^{(a)} \sigma_x^{(b)} \in \mathbb{Z}_2, \quad (4)$$

The overlap probability distribution μ can be implicitly defined through the overlap expectations, that involve both the thermal average and the average E over disorder. For any smooth function F define

$$\langle\langle F(q_x^{(12)}, q_y^{(23)}, \dots) \rangle\rangle = E \left(\langle F(q_x^{(12)}, q_y^{(23)}, \dots) \rangle \right), \quad (5)$$

where expectation with respect to the μ distribution is denoted by $\langle\langle \cdot \rangle\rangle$. All physical observables can be expressed in terms of overlap observables, so that the full physical meaning of these models is contained in the overlap probability measure.

Let us introduce the average over the volume of the site overlap, the total overlap, often simply referred to as “overlap”,

$$q^{(a,b)} = \frac{1}{|\Lambda|} \sum_x q_x^{(a,b)}. \quad (6)$$

Given 2 replicas, the distribution of the overlap $q^{(1,2)}$ will be denoted by $P^{(2)}$. An other interesting observable is the correlation function of site overlap,

$$C(r) = \frac{1}{|\Lambda|} \sum_x \langle\langle q_x^{(1,2)} q_{x+r}^{(1,2)} \rangle\rangle_c. \quad (7)$$

3 The renormalization group transformation

Two main difficulties arise in the application of RG analysis to spin glasses: the choice of the coarse graining transformation and a correct parameterization of the space of Boltzmann measures — that is, the nature of the additional interactions. Let us begin by the second. In spin glasses the single interactions are not set explicitly, but only through their distribution. Instead of coupling constants we have distributions of couplings, and different distributions give rise to different physical models.

Therefore, our proposal is to keep the form of the Hamiltonian fixed, but to take into account a large space of disorder distributions, such that the J 's are not independent variables. This choice is coherent with the random nature of the couplings, and introduces effective long range interactions between the spins through the correlations of the J 's. More generally, new explicit spin interactions could be added, carried by new disorder variables. This procedure would lead to a very large space of models that can be very interesting to be studied, but in our opinion it is unnecessary for our present purpose. Notice that it would be redundant to introduce new spin interactions depending only on the old disorder variables.

The disorder distribution must respect gauge invariance, therefore it will be parameterized as follows

$$\rho(J) = \exp \left(\sum_i K_i W_i \right), \quad (8)$$

where the $K_i \in \mathbb{R}$ are parameters and the W_i are the Wilson's loops, *i.e.*, products of J 's along closed paths. Up to order J^4 , the distribution (8) is of the form

$$\rho_K = C_K \exp \left[\sum_{\langle x, y \rangle} (K_1 J_{xy}^2 + K_2 J_{xy}^4) + K_3 \sum_{\alpha} \square_{\alpha} + o(J^4) \right], \quad (9)$$

where $C_K \in \mathbb{R}$ is a normalization constant, and the symbol \square denotes the plaquette terms of the kind $J_{x,y} J_{y,z} J_{z,w} J_{w,x}$. Expectation with respect to this distribution will be denoted by E_K . The Gaussian Edwards-Anderson model corresponds to $K_1 = -1/2$, $K_i = 0 \forall i \neq 1$, while the \mathbb{Z}_2 , E-A model is obtained in the limit

$$K_1, |K_2| \rightarrow \infty, \text{ s.t. } K_1/K_2 = -2, \quad (10)$$

$$K_i = 0 \quad \forall i > 2.$$

Another interesting distribution is obtained from the \mathbb{Z}_2 E-A model, adding a plaquette term:

$$K_1, |K_2| \rightarrow \infty, \text{ s.t. } K_1/K_2 = -2, \quad (11)$$

$$K_3 \neq 0,$$

$$K_i = 0 \quad \forall i > 3.$$

Such distribution of the disorder variables J 's corresponds to the Boltzmann-Gibbs distribution of the well known pure gauge \mathbb{Z}_2 model [8]. The internal energy of this model is proportional to the average plaquette, that gives a measure of the (short-range) frustration of the spin system. A change of behaviour is to be expected at the deconfinement transition. In dimension $d = 4$, the transition is of the first order: at $|K_3| \geq K_3^c = \log(\sqrt{2} + 1)/2 \simeq 0.4407$ the average plaquette jumps to $|E_K(\square)| \sim 1$. In the limit $K_3 \rightarrow \infty$, frustration disappears and the Ising model is obtained, up to a simple gauge transformation. The deconfined phase is known to be characterized by a non-zero value of the Polyakov loops, that can be positive or negative independently in each direction, thus defining 2^d pure phases. The resulting spin model is a gauge-invariant Ising model, with antiperiodic b.c.'s along the directions of negative Polyakov loop, and periodic b.c.'s in the remaining directions. The opposite limit, $K_3 \rightarrow -\infty$, leads to the fully frustrated model. However, in the case of the Edwards-Anderson model ($K_3 = 0$), we expect the RG flow to remain in the region $|K_3| < K_3^c$. Indeed, this has been verified numerically.

According to equation (5), we introduce the overlap probability distribution μ_K corresponding to the disorder measure E_K . Let us denote by \mathcal{M} the space of such distributions. Notice that the dependence on $\beta = T^{-1}$ of μ_K could be included in the K 's parameters, performing the simple substitution $J \rightarrow T J$, so that the Boltzmann factor does not contain β , and $E_{\beta}(J^2) = \beta^2$.

The choice of the coarse graining transformation is related to the nature of the order parameter. Block transformations of the spins are suitable for the Ising model because the order parameter is the average magnetisation,

and the diverging correlation length is given by the spin correlation function. For spin glasses the elementary physical observable is the site overlap $q_x^{(ab)}$, while the phase transition is characterized by a functional order parameter, the distribution $P^{(2)}$ of the overlap between two replicas. This distribution is non-trivial in the broken replica symmetry phase [4,12]. The correlation length defined by the correlation function $C(r)$ diverges at the critical point. For those reasons, we will perform a block transformation on the site overlap, leading to a renormalized overlap defined by

$$q_B = \text{sign} \left(\sum_{x \in B} q_x \right). \quad (12)$$

The RG transformation R_l for a rescaling factor l is defined on the overlap distribution μ as follows:

$$\mu^{(l)}(q') = \sum_q \hat{R}_l(q', q) \mu(q), \quad (13)$$

$$\hat{R}_l(q', q) = \prod_{x'} \delta \left(q'_{x'}, \text{sign} \left(\sum_{x \in B_l(lx')} q_x \right) \right), \quad (14)$$

where $x' = x/l$ is the rescaled lattice coordinate and $B_l(lx')$ is the block of side l located in $x = lx'$. A block transformation of this kind was also used in [5].

Let us assume the following:

Ansatz: The set \mathcal{M} of all the overlap distributions μ_K that can be represented through the parameters K is invariant under the RG transformations, *i.e.*,

$$R_l(\mu_{K^{(l)}}) = \mu_{K^{(l)}} \in \mathcal{M}. \quad (15)$$

Exploiting the Ansatz, we can introduce the RG transformation on the parameters space

$$K^{(l)} = R_l K^{(1)}, \quad (16)$$

and look for the fixed points $K^* = R_l K^*$ and the corresponding universality classes. The proposition that Gaussian and \mathbb{Z}_2 SG models belong to the same universality class is therefore formulated in a precise theoretical framework. We expect to find the fixed point of the spin glass transition in the deconfined phase, while for $|K_3| > K_3^c$ the phase transition is to be characterized by different fixed points, somehow related to the Ising model (or, more likely, to the diluted Ising model) for K_3 positive, and to the fully frustrated model for $K_3 < -K_3^c$. The relations with the fully frustrated model may be understood, and the intermediate cases would also be clarified, in terms of universality classes.

We have so far introduced a parameterization of the space of overlap probability measures and a renormalization group transformation in this space, suitable for spin glass models. The parameterization is rather implicit, and not easy to be employed in analytic calculations. Therefore, this RG transformation has been tested numerically, with the Monte Carlo method.

4 Finite-size renormalization group

The RG transformation defined in the previous section has been applied to the 4-dimensional E–A spin glass model with periodic b.c., using Monte Carlo method. A typical problem of this approach is that one is limited by the finite size of the lattice to a few RG iterations. Moreover, it is practically very difficult to consider a large number of parameters, giving rise to truncation errors. To minimize the effect of these approximations, the following method has been applied, as introduced in [9]. For a lattice of linear size L , block variables of size $L/2$ are defined, obtaining a 2^d system after one RG iteration. Therefore the number of possible K parameters in the renormalized system is very limited and the truncation error is controllable. By varying the initial lattice size L we obtain many sets of parameters $K'(K; L)$, renormalized by different rescaling factors $l = L/2$, but realized in the same effective lattice 2^d . Therefore we avoided the systematic errors due to the comparison of K 's acting in different effective lattices and, thus, with different truncations. At the critical point, the couplings are independent of the rescaling factor, and the $K'(K_c, L)$, for different L 's, will coincide. This gives an estimate of K_c , *i.e.*, of T_c . For any pair of sizes a different estimate $K_c(L_1, L_2)$ is obtained, with a small dependence on L_1, L_2 vanishing for large sizes. An estimate of the thermal critical index ν can be obtained independently from each coupling, according to the well known formula

$$\nu^{-1} = y_T(K_i) = \frac{\ln \left(\frac{dK_i}{d\beta}(L_1) / \frac{dK_i}{d\beta}(L_2) \right)}{\ln(L_1/L_2)}, \quad (17)$$

Clearly, the numerical values of the resulting fixed point K^* are not the true infinite volume values. However, the estimates of K_c and of the critical index are weakly affected by finite-size effects.

The general idea of interpreting the ratio of different lattice sizes as RG scaling factor was developed by Nightingale and is known as phenomenological RG [10]. In this approach the test quantities are the specific heat or the susceptibility of the original system and not renormalized quantities like in the method exploited in this paper. Moreover, it does not contain any effective Hamiltonian (or effective measure).

In the Monte Carlo runs we have considered the \mathbb{Z}_2 E–A model in 4 dimensions, with periodic boundary conditions. Only one additional parameter, K_3 , has been introduced in the effective measure. The RG transformation used was a majority rule on the block overlap, with a tie-breaker if zero. The renormalized parameters were obtained with a matching condition on the overlap probability distribution. The complete procedure is as follows:

1. MC run on $L = 2$ system, for a wide range of temperatures and values of the K_3 parameter. The configurations of the disorder J were obtained by independent pure-gauge MC runs.
2. MC run on large systems, $L = 6, 8, 10, 12$, measuring the renormalized overlap distribution, for fixed $K_3 = 0$ and various temperatures.

Table 1. Parameters of the spin glass simulations. Columns correspond to the linear size L , the number of MC-PT steps for thermalization and measures, the number of J samples, the number of temperatures for the PT algorithm N_T , temperature step δT and range T_{\min} , T_{\max} .

L	Therm.	Measures	Samples	N_T	δT	T_{\min}	T_{\max}
2	2×10^4	3×10^4	32000	40	0.05	0.9	2.85
6	8×10^4	8×10^4	1024	21	0.025	1.9	2.4
8	10^5	10^5	1024	21	0.025	1.9	2.4
10	10^5	10^5	1024	21	0.025	1.9	2.4
12	10^5	2×10^5	640	27	0.025	1.95	2.6

3. The renormalized overlap distribution is matched to the overlap distribution measured in the $L = 2$ system, finding the effective parameters T' , K'_3 .

5 Numerical results

The algorithm used in the MC runs is a parallel tempering (PT) Metropolis, exploiting multi-spin coding. The fast computer code has been kindly provided by E. Marinari and F. Zuliani. Table 1 lists the parameters used in the MC runs. For the purpose of studying the critical point it is unnecessary to measure observables at very low temperature, and this considerably simplifies the thermalization. Indeed, in the cold phase the renormalized parameters flow rapidly to the zero-temperature fixed point. The lowest temperature attained was about $0.93 T_c$. Thermalization was verified by the symmetry of the $P_J^{(2)}(q)$ for any given sample of J 's, while the efficiency of parallel tempering was controlled through the histogram of the temperatures visited during each run. In the MC runs on the 2^d system large statistics is needed because of the large finite-size fluctuations of both the spin glass and the pure gauge runs. Ten values of K_3 were considered, ranging from -0.06 to -0.28 . The temperature spacing δT is narrow in order to dispose of a dense spanning of temperatures in the subsequent analysis. The overlap distributions of the small system, and the renormalized overlap distributions of the large system have been fitted with polynomials of K_3 and T , respectively, and matched together for many values of T' . This procedure avoids the necessity of bi-dimensional fits. The matching was performed by numerical minimization of the χ^2 . An example of matching of the overlap distributions is shown in Figure 1. Statistical errors on the matching results (the renormalized parameters T' , K'_3) and the physical observables (T_c and ν) were estimated with jack-knife procedure: the results are re-computed many times leaving one (or more) J samples out. The resulting variance is proportional to the variance of the quantity under study (see, *e.g.*, [13]).

The renormalized parameters T' and K'_3 are plotted in Figures 2 and 3. The curves $T'(T; L)$ of Figure 2 cross in a same point, within the error bars, apart from the curve for $L = 6$ which is affected by strong finite size effects. To obtain numerical estimates of the crossing point and

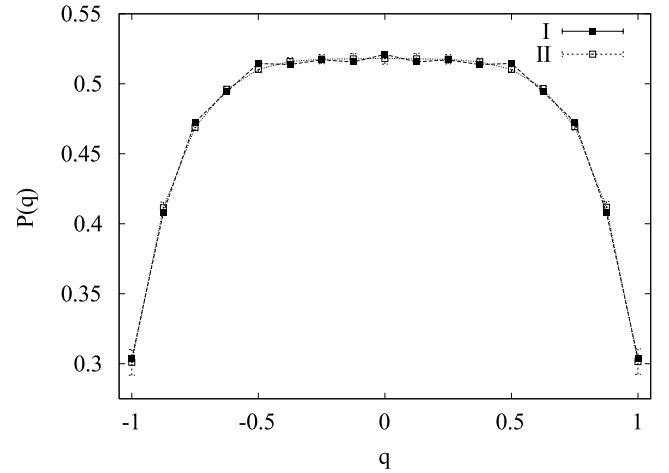


Fig. 1. The overlap distribution of the $L = 2$ system at $T = 1.3$, $K_3 = -0.244(9)$ (I), and the renormalized overlap distribution of the $L = 12$ system at $T = 2.084(4)$ (II), corresponding to the matching point.

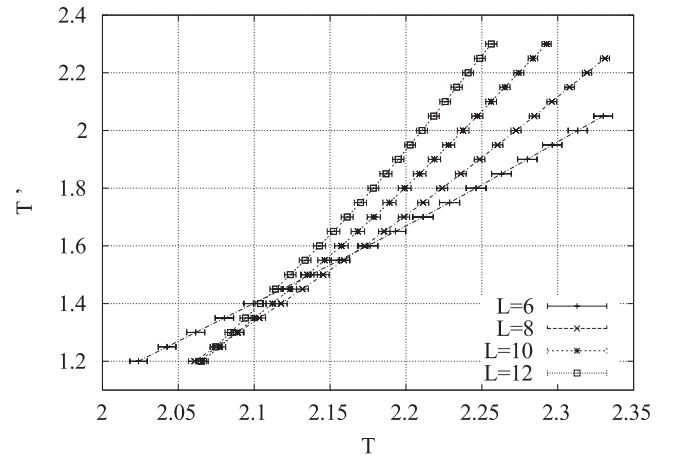


Fig. 2. The renormalized temperature T' as a function of the temperature T .

of the slopes, the curves have been fitted. The results are listed in Table 2. The curves $K_3(T; L)$ of Figure 3 are approximately constant close to the critical temperature, and match for all temperatures for $L = 10, 12$. This means that in the plane (T, K_3) the repulsive direction at the fixed point is orthogonal to the K_3 axis, as sketched in

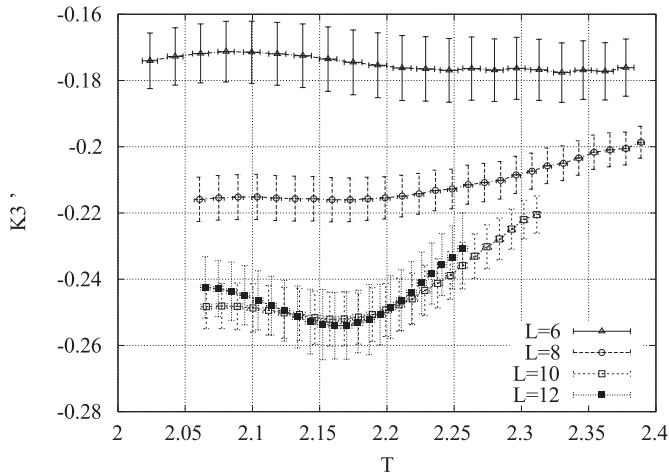


Fig. 3. The renormalized coupling K'_3 as a function of the temperature T . Negative values correspond to negative average plaquette, *i.e.*, increased frustration.

Table 2. Estimates of the critical temperature T_c and exponent ν obtained from the curves of Figure 2.

L/L'	T_c	ν
8/6	2.17(2)	0.87(6)
10/6	2.13(1)	1.01(4)
12/6	2.109(9)	1.05(3)
10/8	2.09(3)	1.1(1)
12/8	2.07(1)	1.07(6)
12/10	2.06(3)	1.1(2)

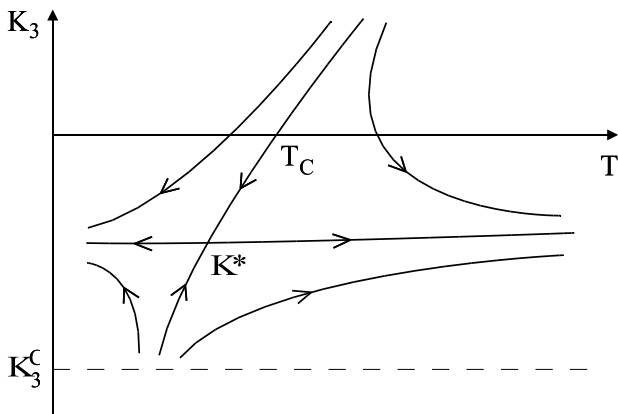


Fig. 4. Qualitative picture of the RG flow showing the finite temperature fixed point. The repulsive direction is approximately horizontal; the critical temperature decreases for larger negative values of K_3 because of the higher frustration.

Figure 4. Estimates of the critical temperature T_c and of the exponent ν agree with previous numerical results by Marinari and Zuliani [11], obtained with the finite-size scaling of Binder cumulant.

6 Conclusions

A renormalization group transformation suitable for spin glasses, and more generally for disordered systems, has been introduced for the first time. The procedure is non-standard for two reasons. 1) The coarse graining transformation is performed on the distribution of site overlaps, which clearly does not have a simple representation of the kind $\exp(-\beta H)$. 2) The space of such distributions is parameterized by additional terms in the distribution of the disorder, while the Hamiltonian of the system is kept fixed. This leads to a quantitative definition of universality classes for spin glass models, clarifying also the connections between \mathbb{Z}_2 and Gaussian spin glasses, the Ising model, the fully frustrated model.

This RG scheme has been tested numerically with Monte Carlo method in the case of the 4-dimensional EA spin glass. Good estimates of the critical temperature and of the exponent ν have been obtained with moderate computer time.

A plot of the RG flow in spin glass models has been sketched for the first time.

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